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Symmetries and supersymmetries of the quantum harmonic oscillator

J Beckers, D Dehin and V Hussin†

Physique théorique et mathématique, Institut de Physique au Sart Tilman, B5, Université de Liège, B-4000 Liège, Belgium

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Abstract. The supersymmetric version of the one-dimensional harmonic oscillator is studied by taking into account its conformal properties. The largest superalgebra of symmetries and supersymmetries is derived as Osp(2/2)□ Sh(1), the semidirect sum of Osp(2/2) and the Heisenberg superalgebra. Through a one-to-one correspondence between the non-relativistic free case and the harmonic oscillator description, we deduce the (expected) supersymmetries of the Schrödinger equation. The above structure appears as the largest spectrum-generating superalgebra of the harmonic oscillator and its representation within an energy basis is given. The physical three-dimensional case is also considered when the maximal set of (super)symmetries is required and this case is compared with recent work.

1. Introduction

Supersymmetric quantum mechanics as initiated by Witten (1981, 1982) has already been developed in much recent work (Salomonson and Van Holten 1982, De Crombrugghe and Rittenberg 1983, Fubini and Rabinovici 1984, Ravndal 1984, D'Hoker and Vinet 1984a, b, 1985a, b, Balantekin 1985, Gamboa and Zanelli 1985 and references therein). More particularly (Witten 1981, 1982, Salomonson and Van Holten 1982, De Crombrugghe and Rittenberg 1983, Ravndal 1984, Balantekin 1985, Gamboa and Zanelli 1985) the supersymmetric harmonic oscillator has been considered as one of the simplest examples in $N=2$ supersymmetric quantum mechanics. In fact, this application is a well defined soluble model which possesses both conformal and supersymmetric invariances like the ones considered by Fubini and Rabinovici (1984) and D'Hoker and Vinet (1984a, 1985a, b). Indeed it is well known (Hagen 1972, Niederer 1972) that the maximal kinematical invariance group of the three-dimensional free Schrödinger equation is a twelve-parameter Lie group (containing the Galilei group, dilations and expansions) and that this invariance group is isomorphic (Niederer 1973) to the largest group (of coordinate transformations) leaving invariant the corresponding Schrödinger equation for the harmonic oscillator. Such a group has been recently (Beckers and Hussin 1984, Hussin and Sinzinkayo 1985, Hussin and Jacques 1986) revisited in connection with its maximal subgroups like SO(3)⊗SO(2, 1) which will be of specific interest here as it was in the study of (super)symmetries of the magnetic monopole (Jackiw 1980, D'Hoker and Vinet 1984a, 1985a). These contributions (Hagen 1972, Niederer 1972, 1973) have opened the so-called 'non-relativistic

† Chargé de recherches FNRS, Belgium.

conformal quantum mechanics' where, in fact, the one-parameter Galilei subgroup of time translations is replaced by the three-parameter group $SL(2,\mathbb{R}) \sim SO(2,1) \sim SU(1,1)$ of (projective) transformations.

So superconformal quantum mechanics deals with the harmonic oscillator: in particular, its superconformal constants of motion can be derived. Moreover the spectrum-generating algebra (Niederer 1973) $ho(1) \approx \widetilde{schr}(1)$ of the harmonic oscillator can lead to an interesting superalgebra as in other contexts already discussed for the magnetic monopole (D'Hoker and Vinet 1984a, 1985a) and other supersymmetric quantum mechanical systems (Balantekin 1985). All these comments do motivate a more complete study of the symmetries and supersymmetries of the harmonic oscillator, which is the aim of this paper. In fact we will insist on the one-dimensional case (§§ 2-5) and we will discuss the three-dimensional case (§§ 6, 7) in connection with recent results (D'Hoker and Vinet 1985a, Balantekin 1985).

In § 2 we just recall a few elements on $N=2$ supersymmetric quantum mechanics (Witten 1981, 1982, Fubini and Rabinovici 1984) and give the known results for the one-dimensional harmonic oscillator. Section 3 is devoted to the *complete* study of the conformal symmetries of the supersymmetric harmonic oscillator as well as to the one-to-one correspondence with the free case. The whole set of supersymmetries is then determined in § 4 leading to the superalgebra Osp(2/2)□Sh(1), this semidirect sum dealing with the so-called Heisenberg superalgebra Sh(1). Then in § 5 we get the oscillator representation of this superalgebra when an energy basis is explicitly used, the generators being physically interpreted as raising or lowering operators. All these results characterise the *largest* superalgebra of the one-dimensional harmonic oscillator. The particularly interesting three-dimensional case is considered in § 6 when we require the maximal set of (super)symmetries in correspondence with the previous sections and a matrix realisation is proposed. Finally in § 7 we compare the above three-dimensional context with Balantekin's (Balantekin 1985) and we adapt D'Hoker and Vinet's construction (D'Hoker and Vinet 1985a) in the context of the harmonic oscillator.

2. $N=2$ supersymmetric quantum mechanics and the one-dimensional harmonic oscillator

Several standard procedures (Witten 1981, 1982, Salomonson and Van Holten 1982, De Crombrugghe and Rittenberg 1983, Fubini and Rabinovici 1984) for introducing supersymmetries have already been discussed. After Witten (1981, 1982) if there are operators $Q^a (a = 1, 2, \ldots, N)$ commuting with the Hamiltonian H

$$[Q^a, H] = 0 \tag{2.1}$$

and satisfying the anticommutation relations

$$\{Q^a, Q^b\} = \delta^{ab} H \tag{2.2}$$

then the quantum mechanical system is supersymmetric.

For a spin-$\frac{1}{2}$ particle moving on the line—the simplest $N=2$ system—we can define two operators associated with the degrees of freedom, i.e.

$$Q_{\pm} = \frac{1}{\sqrt{2}}(Q^1 \pm \mathrm{i} Q^2). \tag{2.3}$$

They satisfy the algebra (in correspondence with (2.1) and (2.2))

$$[Q_\pm, H] = 0 \tag{2.4}$$

$$\{Q_+, Q_-\} = H \qquad \text{and} \qquad \{Q_\pm, Q_\pm\} = 0. \tag{2.5}$$

Such a model in potential theory–supersymmetric quantum mechanics leads to

$$Q_\pm = \frac{1}{\sqrt{2}}\left(p \mp \mathrm{i}\frac{\mathrm{d}W}{\mathrm{d}x}(x)\right)\xi_\pm \tag{2.6}$$

where as usual

$$p = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} \qquad [p, x] = -\mathrm{i} \tag{2.7}$$

$$\{\xi_+, \xi_-\} = 1 \qquad \{\xi_\pm, \xi_\pm\} = 0 \qquad \xi_+^\dagger = \xi_- \tag{2.8}$$

and where the superpotential $W(x)$ is any function of x.

Moreover, if conformal invariance and supersymmetry have to be combined as already discussed by Fubini and Rabinovici (1984) and D'Hoker and Vinet (1984a, b, 1985a, b), we know that the system has a richer algebraic structure containing S-type operators (say S_+ and S_-) besides Q-type operators (defined here by (2.3) in general or by (2.6) in connection with the presence of a superpotential). We will come back to these fermionic generators within the physical context of the supersymmetric harmonic oscillator.

At the moment let us pay attention to the Q and let us notice that we get from (2.5) and (2.6)

$$H = \frac{1}{2}\left[p^2 + \left(\frac{\mathrm{d}W(x)}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}^2W(x)}{\mathrm{d}x^2}\right)[\xi_+, \xi_-]\right]. \tag{2.9}$$

According to (2.3), the Hamiltonian (2.9) leads to

$$Q^1 = \frac{1}{\sqrt{2}}(Q_+ + Q_-) = \frac{1}{2}\left(p\varphi^1 + \frac{\mathrm{d}W(x)}{\mathrm{d}x}\varphi^2\right) \tag{2.10}$$

and

$$Q^2 = \frac{\mathrm{i}}{\sqrt{2}}(Q_- - Q_+) = \frac{1}{2}\left(p\varphi^2 - \frac{\mathrm{d}W(x)}{\mathrm{d}x}\varphi^1\right) \tag{2.11}$$

where the $\varphi^a (a = 1, 2)$ are

$$\varphi^1 = \xi_+ + \xi_- \qquad \varphi^2 = \mathrm{i}(\xi_- - \xi_+) \tag{2.12}$$

and generate a Clifford algebra

$$\{\varphi^a, \varphi^b\} = 2\delta^{ab}\mathbb{1}. \tag{2.13}$$

The latter quantities can easily be realised through 2×2 Pauli matrices

$$\varphi^1 = \sigma^1 \qquad \varphi^2 = \sigma^2 \qquad \xi_+ = \sigma_+ = \tfrac{1}{2}(\sigma^1 + \mathrm{i}\sigma^2) \qquad \xi_- = \sigma_- = \tfrac{1}{2}(\sigma^1 - \mathrm{i}\sigma^2) \tag{2.14}$$

leading to

$$[\xi_+, \xi_-] = \sigma^3 \tag{2.15}$$

and to the Hamiltonian

$$H=\frac{1}{2}\left[p^2+\left(\frac{\mathrm{d}W}{\mathrm{d}x}\right)^2+\frac{\mathrm{d}^2W}{\mathrm{d}x^2}\sigma^3\right]. \tag{2.16}$$

If different choices of superpotentials have already been considered, let us come back here to the very interesting physical case of the harmonic oscillator (the mass m is taken as unity) corresponding to

$$W(x)=\tfrac{1}{2}\omega x^2. \tag{2.17}$$

In such a context, the Hamiltonian (2.16) evidently becomes

$$H=\tfrac{1}{2}(p^2+\omega^2x^2)+\tfrac{1}{2}\omega\sigma^3=H_{\mathrm{B}}+H_{\mathrm{F}} \tag{2.18}$$

where we identify the bosonic and fermionic (expected) parts (Ravndal 1984). The supersymmetric charges are explicitly given by

$$Q_+=\frac{1}{\sqrt{2}}(p-\mathrm{i}\omega x)\begin{pmatrix}0&1\\0&0\end{pmatrix}\qquad Q_-=\frac{1}{\sqrt{2}}(p+\mathrm{i}\omega x)\begin{pmatrix}0&0\\1&0\end{pmatrix} \tag{2.19}$$

or correspondingly

$$Q^1=\tfrac{1}{2}(p\sigma^1+\omega x\sigma^2)\qquad Q^2=\tfrac{1}{2}(p\sigma^2-\omega x\sigma^1). \tag{2.20}$$

3. Conformal symmetries of the supersymmetric harmonic oscillator and the one-to-one correspondence with the free case

The symmetry properties of the n-dimensional harmonic oscillator are well known (Niederer 1973). In the one-dimensional case corresponding to the group HO(1) the associated kinematical invariance algebra is generated by

$$\begin{aligned}
P_+&=K+\mathrm{i}P=\mathrm{i}\exp(-\mathrm{i}\omega t)(p+\mathrm{i}\omega x)=\mathrm{i}(2\omega)^{1/2}\exp(-\mathrm{i}\omega t)a^+\\
P_-&=K-\mathrm{i}P=-\mathrm{i}\exp(\mathrm{i}\omega t)(p-\mathrm{i}\omega x)=-\mathrm{i}(2\omega)^{1/2}\exp(\mathrm{i}\omega t)a\\
C_+&=C_1+\mathrm{i}C_2=\tfrac{1}{2}\mathrm{i}\exp(-2\mathrm{i}\omega t)(p+\mathrm{i}\omega x)^2=\mathrm{i}\omega\exp(-2\mathrm{i}\omega t)(a^+)^2\\
C_-&=C_1-\mathrm{i}C_2=-\tfrac{1}{2}\mathrm{i}\exp(2\mathrm{i}\omega t)(p-\mathrm{i}\omega x)^2=-\mathrm{i}\omega\exp(2\mathrm{i}\omega t)a^2\\
H_{\mathrm{B}}&=\tfrac{1}{2}(p^2+\omega^2x^2)=\tfrac{1}{2}\omega\{a^+,a\}
\end{aligned} \tag{3.1}$$

where the last expressions correspond to the oscillator representation (Niederer 1973) in terms of bosonic creation (a^+) and annihilation (a) operators. Let us notice that the Hermitian operators P, K, C_1, C_2 and H_{B} correspond to the Noetherian constants of motion associated with five types (Niederer 1973) of coordinate transformations such that

$$\delta t=a_0+a_1\sin 2\omega t+a_2\cos 2\omega t \tag{3.2a}$$

$$\delta x=\omega(-a_1\cos 2\omega t+a_2\sin 2\omega t)x+a_3\cos\omega t+a_4\sin\omega t. \tag{3.2b}$$

The non-zero commutation relations are

$$[H_{\mathrm{B}},C_+]=2\omega C_+\qquad [H_{\mathrm{B}},C_-]=-2\omega C_-\qquad [C_+,C_-]=-4\omega H_{\mathrm{B}} \tag{3.3a}$$

$$\begin{aligned}
&[H_{\mathrm{B}},P_+]=\omega P_+\qquad [H_{\mathrm{B}},P_-]=-\omega P_-\\
&[C_+,P_-]=2\mathrm{i}\omega P_+\qquad [C_-,P_+]=2\mathrm{i}\omega P_-
\end{aligned} \tag{3.3b}$$

$$[P_+,P_-]=2\omega I \tag{3.3c}$$

where I is the identity operator. In (3.3) let us point out the so(2, 1) algebra generated by H_B, C_+, C_- (cf (3.3*a*)), these operators being associated with the reparametrisations of time (3.2*a*) and being intimately connected with time translational, dilation and expansion generators included in the algebra Schr(1). Such a subalgebra will give the *conformal* character to our developments as already noticed in other papers (Fubini and Rabinovici 1984, D'Hoker and Vinet 1984a, b, 1985a, b, Balantekin 1985) dealing with supersymmetric quantum mechanics. Moreover let us notice in (3.3) the presence of the Heisenberg algebra h(1) generated by P_+, P_- and I (cf (3.3*c*)). It is in a one-to-one correspondence with the Heisenberg algebra of $\widetilde{\mathrm{Schr}(1)}$, the central extension of Schr(1), generated by the space translational, pure Galilean and identity operators. Consequently the maximal kinematical invariance algebra is the semidirect sum so(2, 1) $\square$ h(1).

Now the *supersymmetric* version of the harmonic oscillator characterised by the Hamiltonian (2.18) again admits all the preceding 'bosonic' symmetries but there are also 'fermionic' ones which are associated with the following operators

$$H_F = \tfrac{1}{2}\omega\sigma_3 \qquad T_\pm = T_1 \pm \mathrm{i}T_2 = \exp(\mp \mathrm{i}\omega t)\sigma_\pm \tag{3.4}$$

the Hermitian operators H_F, T_1, T_2 leading to three Noetherian constants of motion as discussed hereafter. The generators H_F, $T_\pm$ commute with all the operators (3.1) and between themselves they satisfy

$$[H_F, T_\pm] = \pm\omega T_\pm \tag{3.5}$$

and

$$\{T_+, T_-\} = I. \tag{3.6}$$

Indeed all the generators (3.1) (of bosonic type) and H_F (of fermionic type) are even (Rittenberg 1978) and then satisfy only commutation relations while $T_\pm$ (of fermionic type) are odd (Rittenberg 1978) and satisfy both commutation and anticommutation relations. Consequently the supersymmetric version of the one-dimensional harmonic oscillator admits a *superalgebra of symmetries* of dimension 8 (plus 1 for the identity operator). Its supersubalgebra (3.5) and (3.6) has never been exploited to our knowledge (up to recent information communicated to us by Durand (1985)).

All the symmetries (3.1) and (3.4) can be determined within the Lagrangian formalism. The classical Lagrangian associated with the Hamiltonian (2.18) is (Ravndal 1984)

$$L = \tfrac{1}{2}(p^2 - \omega^2 x^2) + \mathrm{i}\Psi^\dagger\dot{\Psi} - \omega\Psi^\dagger\Psi \tag{3.7}$$

where $p = \dot{x}$ and $\Psi = \Psi(t)$. The fermionic variables Ψ and $\Psi^\dagger$ are generators of a Grassmann algebra and the Hamiltonian (2.18) is recovered by making the identification

$$\Psi(t) = \exp(-\mathrm{i}\omega t)\sigma_-. \tag{3.8}$$

It is then easy to prove that the symmetries (3.1) are effectively associated with the coordinate transformations (3.2) with complementary transformations on Ψ given by

$$\delta\Psi = \mathrm{i}\omega\delta t\Psi \tag{3.9}$$

when, in fact, $\Psi(t) \to \Psi'(t') = \delta t\dot{\Psi} + \delta\Psi$. The other remaining symmetries (3.4) correspond only to transformations on the fermionic variables such that

$$\delta\Psi = \mathrm{i}b\Psi + (\mathrm{i}\beta_1 + \beta_2)\exp(-\mathrm{i}\omega t) \tag{3.10}$$

where b is a real parameter while β_1 and β_2 are two real Grassmann parameters. Such transformations lead to $H_F(b)$, $T_1(\beta_1)$ and $T_2(\beta_2)$ as Noetherian constants of motion.

On the basis of the one-to-one correspondence already known (Niederer 1973) between the current bosonic harmonic oscillator and the free case, let us now extend it to the supersymmetric context characterised by the Hamiltonian (2.18). The Schrödinger equation for the supersymmetric harmonic oscillator is

$$(\mathrm{i}\partial_t + \tfrac{1}{2}\partial_x^2 - \tfrac{1}{2}\omega^2 x^2 - \tfrac{1}{2}\omega\sigma_3)\psi(t, x) = 0 \tag{3.11}$$

where ψ is a two-component wavefunction

$$\psi(t, x) = \begin{pmatrix} \psi^{(1)}(t, x) \\ \psi^{(-1)}(t, x) \end{pmatrix}. \tag{3.12}$$

This system can equivalently be written as a set of two equations

$$(\mathrm{i}\partial_t + \tfrac{1}{2}\partial_x^2 - \tfrac{1}{2}\omega^2 x^2 - \tfrac{1}{2}\omega\varepsilon)\psi^{(\varepsilon)}(t, x) = 0 \tag{3.13}$$

for $\varepsilon = \pm 1$. Then let us study the correspondence between (3.13) and the free Schrödinger equation

$$(\mathrm{i}\partial_u + \tfrac{1}{2}\partial_v^2)\varphi(u, v) = 0 \tag{3.14}$$

expressed in terms of oscillator coordinates (Niederer 1973)

$$u = \omega^{-1} \tan \omega t \qquad v = (\cos \omega t)^{-1} x. \tag{3.15}$$

As already noticed (Niederer 1973) in the bosonic context, (3.14) and the equation

$$(\mathrm{i}\partial_t + \tfrac{1}{2}\partial_x^2 - \tfrac{1}{2}\omega^2 x)\chi(t, x) = 0 \tag{3.16}$$

are in such a one-to-one correspondence if

$$\chi(t(u, v), x(u, v)) = (1 + \omega^2 u^2)^{1/4} \exp[-\mathrm{i}\omega^2 u v^2 / 2(1 + \omega^2 u^2)]\varphi(u, v) \tag{3.17}$$

or conversely

$$\varphi(u(t, x), v(t, x)) = (\cos \omega t)^{1/2} \exp(\tfrac{1}{2}\mathrm{i}\omega x^2 \tan \omega t)\chi(t, x). \tag{3.18}$$

Then, in our supersymmetric context, if we take

$$\psi^{(\varepsilon)}(t(u, v), x(u, v)) = f^{(\varepsilon)}(u)\chi(t(u, v), x(u, v)) \tag{3.19}$$

we will get the one-to-one correspondence between (3.13) and (3.14) if and only if $f^{(\varepsilon)}(u)$ satisfies the equation

$$2\mathrm{i}(1 + \omega^2 u^2)\partial_u f^{(\varepsilon)}(u) - \omega\varepsilon f^{(\varepsilon)}(u) = 0. \tag{3.20}$$

This implies up to a constant

$$f^{(\varepsilon)}(u) = \exp(-\tfrac{1}{2}\mathrm{i}\varepsilon \tan^{-1} \omega u). \tag{3.21}$$

Equation (3.19) is finally

$$\psi^{(\varepsilon)}(t(u, v), x(u, v)) = h^{(\varepsilon)}(u, v)\varphi(u, v) \tag{3.22}$$

with

$$h^{(\varepsilon)}(u, v) = (1 + \omega^2 u^2)^{1/4} \exp(-\tfrac{1}{2}\mathrm{i}\varepsilon \tan^{-1} \omega u) \exp\left(-\frac{\mathrm{i}\omega u v^2}{(1 + \omega^2 u^2)}\right). \tag{3.23}$$

Conversely, we have

$$\varphi(u(t,x),v(t,x))=h^{(\varepsilon)\prime}(t,x)\psi^{(\varepsilon)}(t,x) \tag{3.24}$$

with

$$\begin{aligned} h^{(\varepsilon)\prime}(t,x)&=(h^{(\varepsilon)})^{-1}(u(t,x),v(t,x)) \\ &=(\cos\omega t)^{1/2}\exp(\tfrac{1}{2}\mathrm{i}\varepsilon\omega t)\exp(\tfrac{1}{2}\mathrm{i}\omega x^2\tan\omega t). \end{aligned} \tag{3.25}$$

In order to confirm the correspondence at the level of the symmetry operators, we have to transform these operators by $h^{(\varepsilon)}\equiv(3.23)$. More precisely, on the solution $\psi(t,x)\equiv(3.12)$ of the supersymmetric harmonic oscillator, we have

$$\psi(t(u,v),x(u,v))=M(u,v)\begin{pmatrix}\varphi(u,v)\\ \varphi(u,v)\end{pmatrix} \tag{3.26}$$

where

$$\begin{aligned} M(u,v)&=(1+\omega^2u^2)^{1/4}\exp\left(-\frac{\mathrm{i}\omega uv^2}{(1+\omega^2u^2)}\right) \\ &\quad\times\begin{pmatrix}\exp(-\frac{1}{2}\mathrm{i}\tan^{-1}\omega u) & 0\\ 0 & \exp(\frac{1}{2}\mathrm{i}\tan^{-1}\omega u)\end{pmatrix}. \end{aligned} \tag{3.27}$$

Then if we denote by X_{HO} a symmetry operator for our harmonic oscillator the corresponding symmetry operator for the free case becomes

$$X^0(u,v)=M^{-1}X_{\mathrm{HO}}(t(u,v),x(u,v))M. \tag{3.28}$$

It is then easy to show that the usual symmetry operators of spatial translation ($P^0=-\mathrm{i}\partial_v$) and pure Galilean transformation ($K^0=-\mathrm{i}u\partial_v-v$) of the free case are recovered as corresponding to P and K obtained from P_+, P_- in (3.1). Otherwise from the fact that

$$H_{\mathrm{B}}=\tfrac{1}{4}\{P_+,P_-\}\qquad C_+=-\tfrac{1}{2}\mathrm{i}P_+^2\qquad C_-=\tfrac{1}{2}\mathrm{i}P_-^2 \tag{3.29}$$

it is easy to determine the corresponding free operators. The free Schrödinger Hamiltonian operator is then obtained as

$$H^0=-\tfrac{1}{2}\partial_v^2=\tfrac{1}{2}(H_{\mathrm{B}}^0+C_2^0) \tag{3.30}$$

while the dilation (D^0) and expansion (A^0) operators are

$$\begin{aligned} D^0&=2uH_{\mathrm{S}}-\mathrm{i}v\partial_v+\tfrac{1}{2}\mathrm{i}=C_1^0/\omega \\ A^0&=u^2H_{\mathrm{S}}+\mathrm{i}uv\partial_v+\tfrac{1}{2}\mathrm{i}u+\tfrac{1}{2}v^2=(H_{\mathrm{B}}^0-C_2^0)/(2\omega^2). \end{aligned} \tag{3.31}$$

Finally, the additional symmetry operators H_{F}, $T_\pm$ are also transformed to give

$$Y=\frac{1}{\omega}H_{\mathrm{F}}^0=\frac{1}{\omega}H_{\mathrm{F}}=\tfrac{1}{2}\sigma_3\qquad T_\pm^0=\sigma_\pm. \tag{3.32}$$

4. Supersymmetries of the harmonic oscillator

As already noticed in § 2 the supersymmetric charges Q_+ and Q_- given by (2.19) (or Q^1 and Q^2 given by (2.20)) are intimately connected to the construction of the supersymmetric Hamiltonian (2.18). The existence of other supersymmetric charges denoted by S_+ and S_- (or correspondingly S^1 and S^2) is due (Fubini and Rabinovici 1984) to the conformal invariance of the system under consideration. Thus such supercharges can be introduced here. They are found in the following forms:

$$S_\pm = \frac{1}{\sqrt{2}} \exp(\mp 2\mathrm{i}\omega t)(p \pm \mathrm{i}\omega x)\sigma_\pm \tag{4.1}$$

or

$$S^1 = \tfrac{1}{2}\cos 2\omega t(p\sigma^1 - \omega x\sigma^2) + \tfrac{1}{2}\sin 2\omega t(p\sigma^2 + \omega x\sigma^1)$$

and (4.2)

$$S^2 = \tfrac{1}{2}\cos 2\omega t(p\sigma^2 + \omega x\sigma^1) - \tfrac{1}{2}\sin 2\omega t(p\sigma^1 - \omega x\sigma^2).$$

Notice that we have

$$[H, S_\pm] = \pm 2\omega S_\pm \tag{4.3}$$

ensuring

$$(\mathrm{d}/\mathrm{d}t)S_\pm = \partial_t S_\pm + \mathrm{i}[H, S_\pm] = 0.$$

The supersymmetric charges of Q and S types are both associated with supersymmetric coordinate transformations. Within the Lagrangian formalism with $L \equiv (3.7)$ they correspond to

$$\begin{aligned} \delta x &= \frac{\mathrm{i}}{\sqrt{2}}[\mu(t)\Psi^+ + \bar{\mu}(t)\Psi] \\ \delta\Psi &= -\frac{1}{\sqrt{2}}\mu(t)p + \frac{1}{\sqrt{2}}[\partial_t\mu(t) + \mathrm{i}\omega\mu(t)]x \end{aligned} \tag{4.4}$$

where

$$\mu(t) = (\mu_1 + \mathrm{i}\mu_2) + (\nu_1 + \mathrm{i}\nu_2)\exp(-2\mathrm{i}\omega t) \tag{4.5}$$

μ_1, μ_2, ν_1 and ν_2 being real anticommuting parameters associated with Q_1, Q_2, S_1 and S_2 respectively.

Consequently the maximal kinematical invariance superalgebra of the supersymmetric harmonic oscillator is of dimension 12 (plus 1 for the identity operator of the central extension). There are now six odd generators $T_\pm$, $Q_\pm$ and $S_\pm$.

The non-zero commutation and anticommutation relations including the supersymmetric generators are then given by

$$\begin{aligned} &[H_{\mathrm{B}}, Q_\pm] = \mp\omega Q_\pm = -[H_{\mathrm{F}}, Q_\pm] \\ &[H_{\mathrm{B}}, S_\pm] = \pm\omega S_\pm = [H_{\mathrm{F}}, S_\pm] \\ &[C_\pm, Q_\pm] = -2\mathrm{i}\omega S_\pm \\ &[C_\pm, S_\mp] = -2\mathrm{i}\omega Q_\mp \\ &[P_\pm, Q_\pm] = -\sqrt{2}\mathrm{i}\omega T_\pm \\ &[P_\pm, S_\mp] = -\sqrt{2}\mathrm{i}\omega T_\mp \end{aligned} \tag{4.6}$$

and

$$
\begin{aligned}
&\{Q_+, Q_-\} = (H_B + H_F) \qquad \{S_+, S_-\} = (H_B - H_F) \\
&\{Q_\pm, S_\mp\} = \pm \mathrm{i} C_\mp \\
&\{Q_\pm, T_\mp\} = \pm \frac{\mathrm{i}}{\sqrt{2}} P_\mp \\
&\{S_\pm, T_\mp\} = \mp \frac{\mathrm{i}}{\sqrt{2}} P_\pm .
\end{aligned}
\tag{4.7}
$$

Let us notice that the supercharges $Q_\pm$ and $S_\pm$ taken together with H_B, H_F and $C_\pm$ give rise to the superalgebra Osp(2/2) already mentioned by Balantekin (but in the three-dimensional case). Our superalgebra appears as the largest one associated with the supersymmetric one-dimensional harmonic oscillator: besides Osp(2/2) it contains the additional operators $P_\pm$, $T_\pm$ and I generating another superalgebra which could be called the 'Heisenberg superalgebra' Sh(1). In fact we have constructed the semi-direct sum Osp(2/2) $\square$ Sh(1) characterised by the whole set of commutation and anticommutation relations (3.3), (3.5), (3.6), (4.6) and (4.7).

Now in order to understand more deeply the contents of this superalgebra let us show that all the generators of Osp(2/2) are products of the Sh(1) generators. Indeed as we have already noticed the connections between H_B, $C_\pm$ and $P_\pm$ in (3.29), here we can add the following:

$$
H_F = \tfrac{1}{2}\omega [T_+, T_-] \tag{4.8}
$$

and

$$
Q_\pm = \pm \frac{\mathrm{i}}{\sqrt{2}} P_\mp T_\pm \qquad S_\pm = \mp \frac{\mathrm{i}}{\sqrt{2}} P_\pm T_\pm . \tag{4.9}
$$

Finally, in order to complete the one-to-one correspondence mentioned in § 3, we have to add the free generators corresponding to supersymmetry. It is not difficult to show from (4.9) and from the correspondence between $P_\pm$, $T_\pm$ and their free version that

$$
\begin{aligned}
\mathcal{Q}^0_\pm &= \tfrac{1}{2}(Q^0_\pm + S^0_\pm) = \frac{1}{\sqrt{2}} P^0 \sigma_\pm \\
\mathcal{S}^0_\pm &= \pm \frac{\mathrm{i}}{\sqrt{2}} (Q^0_\mp + S^0_\pm) = \frac{1}{\sqrt{2}} K^0 \sigma_\pm .
\end{aligned}
\tag{4.10}
$$

Equations (4.10) give the supersymmetry generators associated with the *free* Schrödinger equation. They are simply expressed in terms of the generators of space translations and pure Galilean transformations and of matrices $\sigma_\pm$. They ensure the expected (Fubini and Rabinovici 1984) anticommutation relations. Through this complete one-to-one correspondence between the supersymmetric harmonic oscillator and the free case we also conclude as expected that the free Schrödinger equation (whose supersymmetric version coincides with the non-supersymmetric one) does admit supersymmetries.

5. The oscillator super-representation on energy states

By using eigenstates of the energy operator, Niederer (1973) has obtained a discrete description of the oscillator representation and has shown that the generators of ho(1) are ladder operators. With respect to such a basis the solutions of (3.16) are labelled by the radial quantum number n and are given by

$$\chi_n(t, x) = c_n \exp[-\mathrm{i}\omega(n+\tfrac{1}{2})t - \tfrac{1}{2}\omega x^2] H_n((2\omega)^{1/2} x) \tag{5.1}$$

where $n = 0, 1, 2, \ldots$, the c_n being *ad hoc* normalisation coefficients and H_n the usual Hermite polynomials.

With a view to the resolution of our equation (3.11) in the supersymmetric context, the two-component solutions have to be

$$\Psi_n^{(\varepsilon)}(t, x) = \alpha^{(\varepsilon)} \exp(-\tfrac{1}{2}\mathrm{i}\varepsilon\omega t)\chi_n(t, x) \qquad \varepsilon = \pm 1. \tag{5.2}$$

The coefficients $\alpha^{(1)}$ and $\alpha^{(-1)}$ are such that

$$(\alpha^{(1)})^2 + (\alpha^{(-1)})^2 = 1 \tag{5.3}$$

in order to ensure, at fixed time, the usual normalisation

$$\int_{-\infty}^{+\infty} \Psi_n^{(\varepsilon)*}(x)\Psi_n^{(\varepsilon)}(x)\,\mathrm{d}x = 1. \tag{5.4}$$

We can then study the action of the generators belonging to the superalgebra pointed out in § 4. From simple and usual quantum mechanical calculations combined with the well known properties of Hermite polynomials we get through (3.1):

$$P_+\Psi_n^{(\varepsilon)}(t, x) = \exp(-\mathrm{i}\omega t)[2\omega(n+1)]^{1/2}\Psi_{n+1}^{(\varepsilon)}(t, x)$$

and (5.5)

$$P_-\Psi_n^{(\varepsilon)}(t, x) = \exp(\mathrm{i}\omega t)(2\omega n)^{1/2}\Psi_{n-1}^{(\varepsilon)}(t, x).$$

Moreover by using (3.1) (or (3.29) in a simpler way) we also obtain

$$C_+\Psi_n^{(\varepsilon)}(t, x) = -\mathrm{i}\omega \exp(-2\mathrm{i}\omega t)[(n+1)(n+2)]^{1/2}\Psi_{n+2}^{(\varepsilon)}(t, x)$$

$$C_-\Psi_n^{(\varepsilon)}(t, x) = \mathrm{i}\omega \exp(2\mathrm{i}\omega t)[n(n-1)]^{1/2}\Psi_{n-2}^{(\varepsilon)}(t, x) \tag{5.6}$$

and

$$H_{\mathrm{B}}\Psi_n^{(\varepsilon)}(t, x) = \omega(n+\tfrac{1}{2})\Psi_n^{(\varepsilon)}(t, x).$$

From the definitions (3.4), we easily obtain

$$T_+\begin{pmatrix}\psi_n^{(1)}\\ \psi_n^{(-1)}\end{pmatrix} = \exp(-\mathrm{i}\omega t)\begin{pmatrix}\psi_n^{(-1)}\\ 0\end{pmatrix} \qquad T_-\begin{pmatrix}\psi_n^{(1)}\\ \psi_n^{(-1)}\end{pmatrix} = \exp(\mathrm{i}\omega t)\begin{pmatrix}0\\ \psi_n^{(1)}\end{pmatrix}. \tag{5.7}$$

Finally the connections (4.8) and (4.9) and the above results give

$$H_{\mathrm{F}}\begin{pmatrix}\psi_n^{(1)}\\ \psi_n^{(-1)}\end{pmatrix} = \frac{\omega}{2}\begin{pmatrix}\psi_n^{(1)}\\ -\psi_n^{(-1)}\end{pmatrix} \tag{5.8}$$

$$Q_+\begin{pmatrix}\psi_n^{(1)}\\ \psi_n^{(-1)}\end{pmatrix} = \mathrm{i}(n\omega)^{1/2}\begin{pmatrix}\psi_{n-1}^{(-1)}\\ 0\end{pmatrix} \qquad Q_-\begin{pmatrix}\psi_n^{(1)}\\ \psi_n^{(-1)}\end{pmatrix} = -\mathrm{i}[(n+1)\omega]^{1/2}\begin{pmatrix}0\\ \psi_{n+1}^{(1)}\end{pmatrix}$$

$$S_+\begin{pmatrix}\psi_n^{(1)}\\ \psi_n^{(-1)}\end{pmatrix} = -\mathrm{i}[(n+1)\omega]^{1/2}\exp(-2\mathrm{i}\omega t)\begin{pmatrix}\psi_{n+1}^{(-1)}\\ 0\end{pmatrix} \tag{5.9}$$

$$S_-\begin{pmatrix}\psi_n^{(1)}\\ \psi_n^{(-1)}\end{pmatrix} = \mathrm{i}(n\omega)^{1/2}\exp(2\mathrm{i}\omega t)\begin{pmatrix}0\\ \psi_{n-1}^{(1)}\end{pmatrix}.$$

Equations (5.5)-(5.9) completely determine the super-representation of the harmonic oscillator associated with the superalgebra Osp(2/2) □ Sh(1). We evidently recognise in (5.5) and (5.6) Niederer's results in the non-supersymmetric context and in particular his one-step and two-step ladder operators (up to signs and factors of i we have the correspondence $C_\pm$, H_B, $P_\pm$ with his notation (Niederer 1973) $I_\pm$, I_3, $P_\pm$, respectively). Our specificities are evidently given by (5.7) and (5.9) showing the action of the generators $T_\pm$, $Q_\pm$ and $S_\pm$ on the eigenfunctions of the orthonormal energy basis. As already noticed (Witten 1981, Salomonson and Van Holten 1982) the ground state has to be annihilated by Q_+ and Q_-. It is precisely given by

$$\psi_0^T = (0, \psi_0^{(-1)}) \simeq (0, \exp(-\omega x^2/2)).$$

Useful information can then be obtained from such results. For example we easily obtain from (5.6) that

$$C_+C_-\Psi_n^{(\varepsilon)} = \omega^2 n(n-1)\Psi_n^{(\varepsilon)} \qquad C_-C_+\Psi_n^{(\varepsilon)} = \omega^2(n+1)(n+2)\Psi_n^{(\varepsilon)}$$

and

$$H_B^2\Psi_n^{(\varepsilon)} = \omega^2(n^2+n+\tfrac{1}{4})\Psi_n^{(\varepsilon)}.$$

Consequently the Casimir operator of the subalgebra so(2, 1) given by

$$C_0 = (1/8\omega^2)(\{C_+, C_-\} - 2H_B^2) \tag{5.10}$$

has the (expected) eigenvalue $\frac{3}{16}$. This subalgebra so(2, 1) previously called 'the spectrum-generating algebra of the harmonic oscillator' is thus incorporated in our superalgebra Osp(2/2) □ Sh(1) as it was in the maximal kinematical invariance algebra (Niederer 1973). Due to the one-step ladder operators P_+ and P_- and to the relations (3.21) and (4.9), we need only one irreducible super-representation of our superalgebra so that Osp(2/2) □ Sh(1) is really the *largest* spectrum-generating superalgebra of the harmonic oscillator.

In order to complete the study of this spectrum-generating superalgebra let us add a few comments on its Casimir operators and the associated supergroup chain. From recent results (D'Hoker and Vinet 1985b, Balentekin 1985) the Osp(2/2) quadratic Casimir operator takes the form

$$C_2(\mathrm{Osp}(2/2)) = C_0 + (1/4\omega^2)H_F^2 + \tfrac{1}{2}\omega(H_F - Q_+Q_- + S_+S_-) \tag{5.11}$$

(notice the following correspondence with Balantekin's notation (1985): $H_B \leftrightarrow K_0$, $C_\pm \leftrightarrow K_\pm$, $H_F \leftrightarrow Y$, $Q_+ \leftrightarrow V_-$, $Q_- \leftrightarrow W_+$, $S_+ \leftrightarrow V_+$, $S_- \leftrightarrow W_-$). In our realisation it has (also) only zero eigenvalues and leads to non-typical (Pais and Rittenberg 1975, Nahm and Scheunert 1976, Scheunert *et al* 1977) representations. Now if Osp(2/2)□Sh(1) is under study, an elegant way to determine the corresponding Casimir consists in noticing that the extended one-dimensional Schrödinger algebra $\widetilde{\mathrm{Schr}(1)}$ ($\simeq$ho(1)) is denoted by $\bar{S}_{6,11}$ in the Burdet-Paterna-Perrin-Winternitz classification (Burdet *et al* 1978). Its Casimir operator C_0' is then easily obtained from $C_0 \equiv$ (5.10) through the following substitutions:

$$C_\pm \to C_\pm' = C_\pm \pm (\mathrm{i}/\sqrt{2})P_\pm^2 \qquad \text{and} \qquad H_B \to H_B' = H_B - \tfrac{1}{4}\{P_+, P_-\}. \tag{5.12}$$

Thus, due to the inclusions of so(2, 1) in $\widetilde{\mathrm{Schr}(1)}$ and of Osp(2/2) in our largest superalgebra, we directly obtain the Casimir operator of Osp(2/2)□Sh(1) from $C_2 \equiv$ (5.11) through the substitutions (5.12) ($C_0 \to C_0'$) and the following ones:

$$\begin{aligned} H_F &\to H_F' = H_F - (\omega/2)[T_+, T_-] \\ Q_\pm &\to Q_\pm' = Q_\pm \mp (\mathrm{i}/\sqrt{2})P_\mp T_\pm \\ S_\pm &\to S_\pm' = S_\pm \pm (\mathrm{i}/\sqrt{2})P_\pm T_\pm \end{aligned} \tag{5.13}$$

leading once again to vanishing C_2' and correspondingly to non-typical representations. Finally the dynamical symmetries of the harmonic oscillator are associated with the supergroup chain

$$\mathrm{OSp}(2/2)\times\mathrm{SH}(1)\supset\mathrm{OSp}(2/2)\supset\mathrm{OSp}(1/2)\times\mathrm{SO}(2)$$
$$\supset\mathrm{Sp}(2)\otimes\mathrm{SO}(2)\supset\mathrm{SO}(2)\otimes\mathrm{SO}(2) \tag{5.14}$$

all the eigenstates being labelled by the eigenvalues of H_B and H_F.

6. The three-dimensional case with the maximal set of symmetries

If the maximal set of symmetries is required, the particularly interesting three-dimensional case can now be considered by emphasising the constructions of the Hamiltonian (§ 6.1), its superalgebra (§ 6.2) and an 8×8 matrix realisation (§ 6.3). The extension to the n-dimensional case (§ 6.4) is also very briefly mentioned. Then all the considerations developed in § 5 apply here without difficulty.

6.1. Construction of the Hamiltonian

The Hamiltonian of the supersymmetric harmonic oscillator in three dimensions can, by analogy with the one-dimensional case and more particularly with (2.9), be defined in the form $(i, j=1,2,3)$

$$H=\tfrac{1}{2}\{\boldsymbol{p}^2+(\boldsymbol{\nabla} W)^2+(\partial_i\partial_j W)[\xi_+^i, \xi_-^j]\} \tag{6.1}$$

where

$$p_i=-\mathrm{i}\partial/\partial r_i=-\mathrm{i}\partial_i \qquad [p_i, r_j]=-\mathrm{i}\delta_{ij} \tag{6.2}$$

$$\{\xi_+^i, \xi_-^j\}=\delta^{ij} \tag{6.3a}$$

$$\{\xi_\pm^i, \xi_\pm^j\}=0 \qquad (\xi_+^i)^\dagger=\xi_-^i \tag{6.3b}$$

while the superpotential is

$$W=W(\boldsymbol{r})=\tfrac{1}{2}\omega\boldsymbol{r}^2. \tag{6.4}$$

We explicitly have

$$H=\tfrac{1}{2}(\boldsymbol{p}^2+\omega^2\boldsymbol{r}^2)+(\omega/2)[\xi_+^i, \xi_-^i]=H_\mathrm{B}+H_\mathrm{F}. \tag{6.5}$$

The supercharges satisfying (2.4) and (2.5) are then

$$Q_\pm=(1/\sqrt{2})(\boldsymbol{p}\mp\mathrm{i}\boldsymbol{\nabla} W)\cdot\boldsymbol{\xi}_\pm=(1/\sqrt{2})(\boldsymbol{p}\mp\mathrm{i}\omega\boldsymbol{r})\cdot\boldsymbol{\xi}_\pm \tag{6.6}$$

and the corresponding Q^1 and Q^2 are again defined in terms of generators $\varphi_i^a (a=1,2)$. We have

$$\begin{aligned}Q^1&=\tfrac{1}{2}(\boldsymbol{p}\cdot\boldsymbol{\varphi}^1+\omega\boldsymbol{r}\cdot\boldsymbol{\varphi}^2)\\ Q^2&=\tfrac{1}{2}(\boldsymbol{p}\cdot\boldsymbol{\varphi}^2-\omega\boldsymbol{r}\cdot\boldsymbol{\varphi}^1)\end{aligned} \tag{6.7}$$

where the φ_i^a satisfy the Clifford algebra

$$\{\varphi_i^a, \varphi_j^b\}=2\delta^{ab}\delta_{ij}. \tag{6.8}$$

6.2. *The maximal kinematical superalgebra*

The maximal kinematical invariance group of the bosonic harmonic oscillator in three dimensions has also been determined by Niederer (1973). It is the so-called HO(3) group isomorphic to $\widetilde{\mathrm{SCHR}(3)}$, the group of invariance of the three-dimensional free equation. The generators of the corresponding algebra are directly obtained for the one-dimensional case. We then have twelve generators denoted H_B, $C_\pm$, $\boldsymbol{P}_\pm$ and $\boldsymbol{L}$, the angular momentum operators (plus the identity). In particular we have

$$[P_{+i}, P_{-j}] = 2\omega\delta_{ij} I. \tag{6.9}$$

Let us notice that

$$\mathrm{ho}(3) \simeq (\mathrm{so}(2,1) \oplus \mathrm{so}(3)) \square \mathrm{h}_3. \tag{6.10}$$

Now due to the independence of H_F in terms of the coordinates $\boldsymbol{p}$ and $\boldsymbol{r}$, the bosonic symmetries are maintained for the total Hamiltonian (6.5) and we can add to them the symmetries associated with the fermionic part. Indeed by analogy with the one-dimensional case, the generators corresponding to the symmetries of H_F are

$$H_F = (\omega/2)[\xi^i_+, \xi^i_-] \qquad \boldsymbol{T}_\pm = \exp(\mp \mathrm{i}\omega t)\boldsymbol{\xi}_\pm. \tag{6.11}$$

We then obtain a superalgebra of symmetries of dimension 20.

In what concerns supersymmetries, besides the generators of Q type given by (6.6) or (6.7), we have the generators of S type directly generalised from the one-dimensional expressions (4.1), i.e.

$$S_\pm = (1/\sqrt{2}) \exp(\mp 2\mathrm{i}\omega t)(\boldsymbol{p} \pm \mathrm{i}\omega \boldsymbol{r}) \cdot \boldsymbol{\xi}_\pm. \tag{6.12}$$

We again notice in correspondence with (4.9)

$$Q_\pm = \pm(\mathrm{i}/\sqrt{2})\boldsymbol{P}_\mp \cdot \boldsymbol{T}_\pm \qquad S_\pm = \mp(\mathrm{i}/\sqrt{2})\boldsymbol{P}_\pm \cdot \boldsymbol{T}_\pm. \tag{6.13}$$

Then, in order to get a closed superalgebra, we have to consider the total angular momentum including spin as expected:

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} = \boldsymbol{r} \times \boldsymbol{p} + \tfrac{1}{2}\mathrm{i}\boldsymbol{\xi}_+ \times \boldsymbol{\xi}_-. \tag{6.14}$$

Finally we conclude that the maximal kinematical invariance superalgebra of the supersymmetric harmonic oscillator is of dimension 24 and is $[\mathrm{Osp}(2/2) \oplus \mathrm{so}(3)] \square \mathrm{Sh}(3)$.

The essential supplementary commutation relations (with respect to the one-dimensional case) are those dealing with the angular momentum operators $\boldsymbol{J} \equiv (6.14)$. Indeed we have

$$\begin{aligned} &[J_i, J_j] = \mathrm{i}\varepsilon_{ijk} J_k \\ &[J_i, P_{\pm j}] = -\mathrm{i}\varepsilon_{ijk} P_{\pm k} \\ &[J_i, T_{\pm j}] = -\tfrac{1}{2}\mathrm{i}\varepsilon_{ijk} T_{\pm k}. \end{aligned} \tag{6.15}$$

6.3. *A 8×8 matrix realisation*

The generators φ^a_i of the Clifford algebra (6.8) can be realised by 8×8 Hermitian matrices. Let us choose the specific Gunaydin and Gürsey (1973) realisation, for example

$$\begin{aligned} &\Gamma_1 = -\sigma_1 \otimes \sigma_1 \otimes \sigma_2 && \Gamma_2 = -\sigma_1 \otimes \sigma_2 \otimes I \\ &\Gamma_3 = \sigma_1 \otimes \sigma_3 \otimes \sigma_2 && \Gamma_4 = \sigma_2 \otimes \sigma_2 \otimes \sigma_1 \\ &\Gamma_5 = -\sigma_2 \otimes \sigma_2 \otimes \sigma_3 && \Gamma_6 = \sigma_2 \otimes I \otimes \sigma_2 \\ &\Gamma_7 = -\sigma_3 \otimes I \otimes I \end{aligned} \tag{6.16}$$

satisfying indeed

$$\Gamma_A^{\dagger} = \Gamma_A \qquad \{\Gamma_A, \Gamma_B\} = 2\delta_{AB}. \tag{6.17}$$

We can make the correspondence

$$\begin{aligned} &\varphi_1^1 = \Gamma_4 && \varphi_1^2 = \Gamma_1 && \varphi_2^1 = \Gamma_5 \\ &\varphi_2^2 = -\Gamma_2 && \varphi_3^1 = \Gamma_6 && \varphi_3^2 = -\Gamma_3 \end{aligned} \tag{6.18}$$

so that

$$\xi_+^1 = \tfrac{1}{2}(\Gamma_4 + \mathrm{i}\Gamma_1) \qquad \xi_+^2 = \tfrac{1}{2}(\Gamma_5 - \mathrm{i}\Gamma_2) \qquad \xi_+^3 = \tfrac{1}{2}(\Gamma_6 - \mathrm{i}\Gamma_3). \tag{6.19}$$

It is then easy to show that

$$\begin{aligned} [\xi_+^i, \xi_-^i] &= \begin{pmatrix} 2\sigma^3 + \mathbb{1} & & 0 & \\ & -\mathbb{1} & & \\ 0 & & -(2\sigma^3 + \mathbb{1}) & \\ & & & \mathbb{1} \end{pmatrix} \\ &= \mathrm{diag}(3, -1, -1, -1, -3, 1, 1, 1). \end{aligned} \tag{6.20}$$

Within such a representation the Schrödinger equation for the supersymmetric harmonic oscillator is

$$(\mathrm{i}\partial_t + \tfrac{1}{2}\Delta - \tfrac{1}{2}\omega^2 \boldsymbol{r}^2 - \tfrac{1}{2}\omega[\xi_+^i, \xi_-^i])\psi(t, \boldsymbol{r}) = 0 \tag{6.21}$$

where ψ is an eight-component wavefunction. In the energy basis corresponding to the developments of § 5 but in the three-dimensional context, we get the eight components

$$\begin{aligned} &\psi_n^1(t, \boldsymbol{r}) = \alpha_1 \exp(-3\mathrm{i}\omega t/2)\chi_n(t, \boldsymbol{r}) \\ &\psi_n^k(t, \boldsymbol{r}) = \alpha_k \exp(\mathrm{i}\omega t/2)\chi_n(t, \boldsymbol{r}) \qquad k = 2, 3, 4 \\ &\psi_n^5(t, \boldsymbol{r}) = \alpha_5 \exp(3\mathrm{i}\omega t/2)\chi_n(t, \boldsymbol{r}) \\ &\psi_n^l(t, \boldsymbol{r}) = \alpha_l \exp(-\mathrm{i}\omega t/2)\chi_n(t, \boldsymbol{r}) \qquad l = 6, 7, 8 \end{aligned} \tag{6.22}$$

where $\chi_n(t, \boldsymbol{r})$ is a solution of the usual harmonic oscillator and where the normalisation condition is

$$(\alpha_1)^2 + \cdots + (\alpha_8)^2 = 1. \tag{6.23}$$

The ground state is then given by taking all the α equal to zero apart from α_5:

$$\psi_0^T(t, \boldsymbol{r}) = (0, 0, 0, 0, \psi_0^5(\boldsymbol{r}), 0, 0, 0). \tag{6.24}$$

6.4. *Remarks on the n-dimensional case*

The above procedure can easily be extended to the n-dimensional case in order to describe the supersymmetric n-dimensional harmonic oscillator admitting a maximal number of additional 'fermionic' symmetries and supersymmetries compatible with those of the non-supersymmetric one. Indeed the maximal kinematical invariance algebra $\mathrm{ho}(n) \simeq (\mathrm{so}(2,1) \oplus \mathrm{so}(n)) \square \mathrm{h}(n)$ of dimension $\{3+[n(n-1)/2]+2n+1\}$ can be enlarged to a (maximal kinematical invariance) superalgebra $(\mathrm{Osp}(2/2) \oplus \mathrm{so}(n)) \square \mathrm{Sh}(n)$ of dimension $\{8+[n(n-1)/2]+4n+1\}$.

Let us notice that the particular two-dimensional case gives rise to a superalgebra of dimension 18 including the so(2) algebra. The algebra (6.3) with $i, j = 1, 2$ is then easily realised by 4×4 matrices (D'Hoker and Vinet 1985b) such that

$$\xi^1_+ = \begin{pmatrix} 0 & -\sigma_+ \\ \sigma_+ & 0 \end{pmatrix} \qquad \xi^2_+ = \frac{1}{2}\begin{pmatrix} 0 & -(\sigma^3+1) \\ \sigma^3 - 1 & 0 \end{pmatrix} \tag{6.25}$$

leading to the Hamiltonian (compare with (6.5))

$$H = \begin{pmatrix} \frac{1}{2}(\boldsymbol{p}^2 + \omega^2 \boldsymbol{x}^2 + 2\omega\sigma^3) & 0 \\ 0 & \frac{1}{2}(\boldsymbol{p}^2 + \omega^2 \boldsymbol{x}^2) \end{pmatrix} \tag{6.26}$$

the ground state being given here by

$$\psi_0^T = (0, \psi_0^2(\boldsymbol{x}), 0, 0).$$

7. Other three-dimensional cases for the supersymmetric harmonic oscillator

In the previous section, we have considered the supersymmetric harmonic oscillator admitting a maximal set of symmetries. This was possible as long as we superpose the bosonic and fermionic parts of the Hamiltonian without specific coupling. Let us here briefly describe some cases with interesting couplings following the contributions of Balantekin (1985) and Gamboa and Zanelli (1985) on the one hand (§ 7.1) and following D'Hoker and Vinet (1985a) on the other hand (§ 7.2). Both cases do evidently contain the usual *bosonic* Hamiltonian in three dimensions.

7.1. *The spin-orbit coupling*

Starting with the supercharges $Q_\pm \equiv (6.6)$, we can construct a supersymmetric Hamiltonian other than (6.1). Indeed, if the conditions (6.3*a*) are replaced by

$$\{\xi^i_+, \xi^j_-\} = \delta^{ij} + \Xi^{ij} \qquad \Xi^{ij} = -\Xi^{ji} \tag{7.1}$$

we get from (2.5) a Hamiltonian which effectively contains the usual bosonic one (H_{B}) but another fermionic part explicitly dependent on $\boldsymbol{r}$ and $\boldsymbol{p}$. Indeed the Hamiltonian (6.6) is now replaced by

$$H = \frac{1}{2}\{\boldsymbol{p}^2 + (\boldsymbol{\nabla} W)\} + \frac{1}{2}(\partial_i \partial_j W)[\xi^i_+, \xi^j_-] - \frac{1}{2}\mathrm{i}[(\partial_i W)p_j - (\partial_j W)p_i]\Xi^{ij} \tag{7.2}$$

and the corresponding supersymmetric harmonic oscillator Hamiltonian then becomes (compare with (6.5))

$$H = \frac{1}{2}(\boldsymbol{p}^2 + \omega^2 \boldsymbol{r}^2) + \frac{1}{2}\omega[\xi^i_+, \xi^i_-] - \frac{1}{2}\mathrm{i}\omega(r_i p_j - r_j p_i)\Xi^{ij}. \tag{7.3}$$

Let us now realise the algebra (7.1) with the conditions (6.3*b*) by 4×4 matrices (Balantekin 1985, Gamboa and Zanelli 1985):

$$\xi_+^i = \sigma^i \otimes \sigma_+ = \begin{pmatrix} 0 & \sigma^i \\ 0 & 0 \end{pmatrix} \qquad \xi_-^i = \sigma^i \otimes \sigma_- = \begin{pmatrix} 0 & 0 \\ \sigma^i & 0 \end{pmatrix}. \tag{7.4}$$

We then obtain

$$\{\xi_+^i, \xi_-^j\} = \delta^{ij}\mathbb{1} + \mathrm{i}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix} \tag{7.5}$$

implying that

$$\Xi^{ij} = \mathrm{i}\varepsilon^{ijk}\sigma^k \otimes \sigma^3 = \mathrm{i}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix} \tag{7.6}$$

and

$$[\xi_+^i, \xi_-^j] = \delta^{ij}(\mathbb{1}\otimes\sigma^3) + \mathrm{i}\varepsilon^{ijk}(\sigma^k \otimes \mathbb{1}) = \delta^{ij}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} + \mathrm{i}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \tag{7.7}$$

Finally the explicit Hamiltonian (7.3) is

$$H = \tfrac{1}{2}(\boldsymbol{p}^2 + \omega^2 \boldsymbol{r}^2)\mathbb{1} + \omega(\boldsymbol{\sigma}\cdot\boldsymbol{L} + \tfrac{3}{2}\mathbb{1})\otimes\sigma^3 \tag{7.8}$$

and can be split into two Hamiltonians $H_\pm$ (2×2 matrices)

$$H_\pm = \tfrac{1}{2}(\boldsymbol{p}^2 + \omega^2 \boldsymbol{r}^2)\mathbb{1} \pm \omega(\boldsymbol{\sigma}\cdot\boldsymbol{L} + \tfrac{3}{2}\mathbb{1}). \tag{7.9}$$

By this way we have explained the Hamiltonian proposed by Balantekin (1985) and Ui and Takeda (1984).

Let us recall that Balantekin (1985) has shown that the Hamiltonian (7.8) admits a dynamical supersymmetry associated with the superalgebra $\mathrm{Osp}(2/2)\oplus\mathrm{so}(3)$ identical with ours (see § 6.2). Such a result is immediately recovered here. Indeed, since the *conformal* symmetries (generated by H_{B}, $C_\pm$) of the bosonic part are conserved by the fermionic part (there is a direct sum between the algebra so(2, 1) and the algebra of rotations generated by $\boldsymbol{L}$), the conformal supersymmetries of S type are also conserved. In contrast, here our superalgebra Sh(3) cannot be included into the dynamical supersymmetry algebra. Indeed the generators $\boldsymbol{P}_\pm$ satisfying (6.15) are not associated with conserved quantities. The symmetries of $\boldsymbol{P}_\pm$ type of the bosonic part are broken by the fermionic part. Finally the symmetries of $\boldsymbol{T}_\pm$ type are also broken by the total Hamiltonian (7.8) due to the presence of the spin-orbit coupling term.

7.2. *The position-like coupling*

The procedures described in §§ 6 and 7.1 ask for supercharges of Q type given by (6.6) ensuring the relations (2.4) and (2.5). They only differ by requiring that the fermionic generators $\xi_\pm^i$ satisfy different algebras ((6.3*a*) or (7.1) respectively). Another procedure has also been proposed by D'Hoker and Vinet (1985a) for the determination of a supersymmetric Hamiltonian describing the dynamics of a spinning particle in the presence of a λ^2/r^2 potential. Here let us apply this last method but to the supersymmetric harmonic oscillator.

After D'Hoker and Vinet we can first define the supercharges

$$Q_\pm = \frac{1}{\sqrt{2}}(\boldsymbol{p}\cdot\boldsymbol{\eta}_\pm + V\eta_\pm^0) \qquad V = [(\boldsymbol{\nabla} W)^2]^{1/2}. \tag{7.10}$$

From (2.5) we immediately obtain

$$H=\tfrac{1}{2}[p_ip_j\{\eta^i_+,\eta^j_-\}+(\nabla W)^2\{\eta^0_+,\eta^0_-\}]+\tfrac{1}{2}Vp_i(\{\eta^0_+,\eta^i_-\}+\{\eta^i_+,\eta^0_-\})$$
$$-\tfrac{1}{2}\mathrm{i}(\partial_iV)(\eta^i_+\eta^0_-+\eta^i_-\eta^0_+). \tag{7.11}$$

From the condition that the bosonic part must be recovered we obtain

$$\{\eta^i_+,\eta^j_-\}=\delta^{ij}+\Xi^{ij}\qquad \Xi^{ij}=-\Xi^{ji}$$
$$\{\eta^0_+,\eta^0_-\}=\mathbb{1} \tag{7.12}$$

while from the Hermiticity condition of the Hamiltonian (7.11), we obtain

$$\{\eta^0_+,\eta^i_-\}+\{\eta^i_+,\eta^0_-\}=0. \tag{7.13}$$

Then, for example, we can realise the algebra (7.12) and (7.13) in terms of 4×4 matrices:

$$\eta^i_\pm=\pm\tfrac{1}{2}(1\pm\mathrm{i}\gamma^5)\gamma^i\qquad \eta^0_\pm=\pm\tfrac{1}{2}\mathrm{i}(1\pm\mathrm{i}\gamma^5)\gamma^0 \tag{7.14}$$

where the Dirac matrices $\gamma^\mu(\theta=0,1,2,3)$ satisfy

$$\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}\qquad (\gamma^0)^\dagger=\gamma^0\qquad (\gamma^i)^\dagger=-\gamma^i$$
$$\gamma^5=\gamma^0\gamma^1\gamma^2\gamma^3. \tag{7.15}$$

The conditions (7.12) are then satisfied by

$$\Xi^{ij}=-\tfrac{1}{2}\mathrm{i}\gamma^5[\gamma^i,\gamma^j]. \tag{7.16}$$

The Hamiltonian (7.11) finally becomes

$$H=\tfrac{1}{2}[\boldsymbol{p}^2+(\nabla W)^2]+\tfrac{1}{2}\gamma^0\boldsymbol{\gamma}\cdot\nabla V \tag{7.17}$$

and for the supersymmetric harmonic oscillator we explicitly have

$$H=\tfrac{1}{2}(\boldsymbol{p}^2+\omega^2\boldsymbol{r}^2)+\tfrac{1}{2}\omega\gamma^0\boldsymbol{\gamma}\cdot\hat{\boldsymbol{r}}\qquad (\hat{\boldsymbol{r}}=\boldsymbol{r}/r). \tag{7.18}$$

It is easy to show that the supercharges $Q_\pm\equiv(7.10)$ are conserved quantities, i.e. verify (2.4). Explicitly they become

$$Q_\pm=\pm\frac{1}{2\sqrt{2}}(1\pm\mathrm{i}\gamma^5)(\boldsymbol{\gamma}\cdot\boldsymbol{p}+\mathrm{i}\omega\gamma^0\boldsymbol{r}) \tag{7.19}$$

and in connection with the definitions (2.10) and (2.11) of Q^1 and Q^2, they lead to

$$Q^1=\tfrac{1}{2}\mathrm{i}\gamma^5(\boldsymbol{\gamma}\cdot\boldsymbol{p}+\mathrm{i}\omega\gamma^0\boldsymbol{r})\qquad Q^2=\gamma^5Q^1 \tag{7.20}$$

these expressions being analogous to those given by D'Hoker and Vinet (1985a) in their context.

Unfortunately such a construction leads us to a Hamiltonian (7.18) which does not admit *conformal* (super)symmetries due to the presence of the fermionic part. The only conserved quantities are the supercharges $Q_\pm$ (or Q^1 and Q^2) and the angular momentum

$$\boldsymbol{J}=\boldsymbol{r}\times\boldsymbol{p}+\tfrac{1}{2}\boldsymbol{\Sigma}\qquad \boldsymbol{\Sigma}=\tfrac{1}{2}\mathrm{i}\boldsymbol{\gamma}\times\boldsymbol{\gamma}. \tag{7.21}$$

With the Hamiltonian (7.18), we obtain a six-dimensional superalgebra of invariance such that

$$[H,Q_\pm]=0\qquad \{Q_+,Q_-\}=H$$
$$[H,\boldsymbol{J}]=0\qquad [Q_\pm,\boldsymbol{J}]=0. \tag{7.22}$$

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